THE FORMULATION OF THE RELATIVISTIC STATISTICAL MECHANICS

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A THESIS

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Physics

KANSAS STATE COLLEGE
OF AGRICULTURE AND APPLIED SCIENCE

1957

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INTRODUCTION

Bergman (1) started to generalize Gibb's classical statistical mechanics so as to make it applicable to both relativistic and quantum mechanical system. His interesting method, however, involves some complicated differential geometry and the resulting formula are not easily manageable. Scheidegger and Krotkov (10) tried to avoid such complexity by by-passing the intermediate step, nonquantized relativistic statistical mechanics, and directly formulated the quantized statistical mechanics which led merely to a relativistic expansion of Maxwell-Boltzman distribution law.

In this paper an effort is made to start from a general classical basis and obtain a simpler result. Nonquantized relativistic statistical mechanics is formulated by expanding the classical Maxwell-Boltzman distribution law and the Gibb's statistical mechanics for a system, by requiring conservation of momentum now as well as of energy of an isolated system.

One will find that it is helpful in this discussion to introduce three reference frames; rest, secondary and primary, Leaf (8). The quantized relativistic statistical mechanics can be obtained from this formulation by simply introducing operators for the appropriate physical variables.

REFERENCE FRAMES AND NOTATION

The rest frame is attached momentarily to an element of a physical system whose properties are to be measured. The primary frame with respect to which the rest frame has relative velocity v as measured by the primary observer, is further introduced. However in many physical experiments, the primary observer does not measure the property of the system relative to

the rest frame, but rather relative to some arbitrary frame, the secondary frame (8). For example, consider the concept of temperature in a system of molecules. If a rest frame is fixed at each individual molecule, then clearly the temperature has no significant meaning with respect to these frames. However when a secondary reference frame which is attached to the container of the system is introduced, the concept of temperature will be meaningful for the system as a whole. Indeed in thermometry and in the theory of gases, it is essential to take a standard in the secondary frame and measure the temperature as a proper quantity, since the indication of a thermometer moving rapidly through a fluid is of no practical interest.

Now let the secondary frame have a constant velocity v* with respect to the primary frame as measured by a primary observer and let the rest frame have a velocity v' with respect to the secondary frame as measured by a secondary observer.

The transformation $P'_\sigma = L'_{\sigma\tau} P^\circ_\tau$ gives the values P'_σ of measurements made upon the quantity P°_τ in the rest frame by a secondary observer and $P_\sigma = L^*_{\sigma\tau} P'_\tau$ gives the values P_σ of measurement made upon the quantity P°_τ in the rest frame by a primary observer. In the above relations, the summation convention for τ is used, and unless specified otherwise, this summation convention will be used in the rest of this paper. By requiring the group property of the Lorentz transformation, one has:

$$L_{\sigma\tau} = L^*_{\sigma\rho} L'_{\rho\tau} \tag{1}$$

Further $P^*_\sigma = L^*_{\sigma\tau} P^\triangle_\tau$ gives the value P^*_σ of measurement made by the primary observer upon a property $P^\triangle_\tau$ of a physical system at rest in the secondary frame. By the orthogonality property of the Lorentz transformation

matrix, the Kronecker's delta is given by:

$$\delta^{*}_{z\rho} = L^{*}_{z\sigma} L^{*}_{\rho\sigma} = L^{*}_{\sigma z} L^{*}_{\sigma\rho}$$
$$\delta_{z\rho} = L_{z\sigma} L_{\rho\sigma} = L_{\sigma z} L_{\sigma\rho}, \quad \text{etc.} \tag{2}$$

Unit vectors at rest in the secondary frame will be denoted by:

$$\hat{e}^{\triangle}_{a)\sigma} = (\hat{i}^{\triangle}_{\sigma}, \hat{j}^{\triangle}_{\sigma}, \hat{k}^{\triangle}_{\sigma}, \hat{u}^{\triangle}_{\sigma})$$
$$\hat{i}^{\triangle}_{\sigma} = (1, 0, 0, 0)$$
$$\hat{j}^{\triangle}_{\sigma} = (0, 1, 0, 0) \tag{3}$$
$$\hat{k}^{\triangle}_{\sigma} = (0, 0, 1, 0)$$
$$\hat{u}^{\triangle}_{\sigma} = (0, 0, 0, i)$$

The configuration space in the primary frame can be defined by the orthogonal unit vectors:

$$\hat{e}^{*}_{a)\sigma} = (\hat{i}^{*}_{\sigma}, \hat{j}^{*}_{\sigma}, \hat{k}^{*}_{\sigma}, \hat{u}^{*}_{\sigma})$$
$$\hat{i}^{*}_{\sigma} = L^{*}_{\sigma z} \hat{i}^{\triangle}_{z}, \quad \hat{j}^{*}_{\sigma} = L^{*}_{\sigma z} \hat{j}^{\triangle}_{z} \tag{4}$$
$$\hat{k}^{*}_{\sigma} = L^{*}_{\sigma z} \hat{k}^{\triangle}_{z}, \quad \hat{u}^{*}_{\sigma} = L^{*}_{\sigma z} \hat{u}^{\triangle}_{z}$$

Then the following relation can be verified:

$$\delta^{*}_{\sigma z} = \bar{\delta}^{*}_{\sigma z} - \hat{u}^{*}_{\sigma} \hat{u}^{*}_{z}$$

where
$$\bar{\delta}^{*}_{\sigma z} = \hat{i}^{*}_{\sigma} \hat{i}^{*}_{z} + \hat{j}^{*}_{\sigma} \hat{j}^{*}_{z} + \hat{k}^{*}_{\sigma} \hat{k}^{*}_{z} \tag{5}$$

Now for a given vector P_σ :

$$P_\sigma = \delta^{*}_{\sigma z} P_z = (\bar{\delta}^{*}_{\sigma z} - \hat{u}^{\triangle}_{\sigma} \hat{u}^{\triangle}_{z}) P_z$$
$$= \bar{\delta}^{*}_{\sigma z} P_z - \hat{u}^{*}_{\sigma} \hat{u}^{*}_{z} P_z = P^{(s)}_{\sigma} + P^{(t)}_{\sigma}. \tag{6}$$

where:

$$P^{(s)}_{\sigma} = \bar{\delta}^{*}_{\sigma z} P_z = L^{*}_{\sigma z} \begin{pmatrix} P_1' \\ P_2' \\ P_3' \\ 0 \end{pmatrix}$$

is the value of the space components (P_1', P_2', P_3', 0) relative to the secondary frame as measured by a primary observer.

$$P_\sigma^{(4)} = -\hat{u}_\sigma^{*}\hat{u}_z^{*} P_z = L_{\sigma z}^{*} \begin{pmatrix} 0 \\ 0 \\ 0 \\ P'_4 \end{pmatrix}$$

is the time component $(0, 0, 0, P'_4)$ relative to the secondary frame as measured by the primary observer.

Further unit vectors at rest in each rest frame will be introduced as:

$$\hat{e}^{\circ}_{\alpha]\sigma} = (\hat{i}^{\circ}_\sigma,\ \hat{j}^{\circ}_\sigma,\ \hat{k}^{\circ}_\sigma,\ \hat{u}^{\circ}_\sigma)$$

$$\hat{i}^{\circ}_\sigma = (1, 0, 0, 0), \qquad \hat{j}^{\circ}_\sigma = (0, 1, 0, 0) \tag{7}$$

$$\hat{k}^{\circ}_\sigma = (0, 0, 1, 0), \qquad \hat{k}^{\circ}_\sigma = (0, 0, 0, \ \ )$$

The orthogonal unit vectors in the primary frame being related to the unit vectors $\hat{e}^{\circ}_{\alpha]z}$ in the j-th rest frame:

$$\hat{e}^{(j)}_{\alpha]\sigma} = (\hat{i}^{(j)}_\sigma,\ \hat{j}^{(j)}_\sigma,\ \hat{k}^{(j)}_\sigma,\ \hat{u}^{(j)}_\sigma)$$

can be defined by:

$$\hat{i}^{(j)}_\sigma = L^{(j)}_{\sigma z}\,\hat{i}^{\circ}_z \quad , \qquad \hat{j}^{(j)}_\sigma = L^{(j)}_{\sigma z}\,\hat{j}^{\circ}_z$$

$$\hat{k}^{(j)}_\sigma = L^{(j)}_{\sigma z}\,\hat{k}^{\circ}_z \quad , \qquad \hat{u}^{(j)}_\sigma = L^{(j)}_{\sigma z}\,\hat{u}^{\circ}_z$$

or, in the abbreviated notation, by:

$$\hat{e}^{(j)}_{\alpha]\sigma} = L^{(j)}_{\sigma z}\,\hat{e}^{\circ}_{\alpha]z} \tag{8}$$

where $L^{(j)}_{\sigma z}$ is the Lorentz transformation matrix which connects the j-th rest frame and the primary frame.

Considering a system of particles whose position 4 - vectors relative to the primary frame are given by:

$$x^{(j)}_\sigma = (x^{(j)}_1,\ x^{(j)}_2,\ x^{(j)}_3,\ x^{(j)}_4 = ict^{(j)})$$

with the superscript j refering to the j-th particle, the proper time interval $d\tau^{(j)}$ in the j-th reference frame is defined by:

$$d\tau^{(j)} = \frac{1}{c}\sqrt{-dx^{(j)}_\sigma dx^{(j)}_\sigma} = \sqrt{1-\beta^{(j)2}}\ dt^{(j)} \tag{9}$$

where:
$$\underline{v}^{(j)} = \left(\frac{dx_1^{(j)}}{dt^{(j)}}, \frac{dx_2^{(j)}}{dt^{(j)}}, \frac{dx_3^{(j)}}{dt^{(j)}} \right)$$

$$v^{(j)} = \left[\left(\frac{dx_1^{(j)}}{dt^{(j)}}\right)^2 + \left(\frac{dx_2^{(j)}}{dt^{(j)}}\right)^2 + \left(\frac{dx_3^{(j)}}{dt^{(j)}}\right)^2 \right]^{1/2}$$

and $\beta^{(j)} = v^{(j)}/c$ (j = 1, 2, - - - - - - -)

The velocity four-vector is now defined by:

$$u_\sigma^{(j)} = \frac{dx_\sigma^{(j)}}{d\tau^{(j)}} = \left(\frac{\underline{v}^{(j)}}{\sqrt{1-\beta^{(j)2}}}, \frac{ic}{\sqrt{1-\beta^{(j)2}}} \right) \tag{10}$$

and the energy-momentum 4 - vector by:

$$p_\sigma^{(j)} = m_0^{(j)} u_\sigma^{(j)} = (\underline{p}^{(j)}, iE^{(j)}/c) \tag{11}$$

with

$$\underline{p}^{(j)} = m^{(j)} \underline{v}^{(j)}, \quad E^{(j)} = m^{(j)} c^2, \quad m^{(j)} = \frac{m_0^{(j)}}{\sqrt{1-\beta^{(j)2}}} \tag{12}$$

Now one can define:

$$\xi_1^{(j)} = \hat{i}_\sigma^* x_\sigma^{(j)}, \quad \xi_2^{(j)} = \hat{j}_\sigma^* x_\sigma^{(j)}, \quad \xi_3^{(j)} = \hat{k}_\sigma^* x_\sigma^{(j)} \tag{13}$$

which are equal in magnitude to the Cartesian components of position as measured relative to the secondary frame. Further the invariant quantities which are equal in magnitude to the Cartesian components of momentum of the particles relative to the secondary frame, are defined by:

$$\Pi_1^{(j)} = \hat{i}_\sigma^* p_\sigma^{(j)}, \quad \Pi_2^{(j)} = \hat{j}_\sigma^* p_\sigma^{(j)}, \quad \Pi_3^{(j)} = \hat{k}_\sigma^* p_\sigma^{(j)} \tag{14}$$

(j = 1, 2, - - - - - - -)

In connection with these quantities, one may define the invariant quantities:

$$\Pi_4^{(j)} = -\hat{u}_\sigma^* p_\sigma^{(j)} = -\hat{u}_\sigma^* p_\sigma'^{(j)} = E'^{(j)}/c \tag{15}$$

where $E'^{(j)}$ is equal to the relativistic kinetic energy of the j-th particle relative to the secondary frame.

$$\therefore (\Pi_4^{(j)})^2 = (-\hat{u}_\tau^* p_\tau^{(j)})(-\hat{u}_\sigma^* p_\sigma^{(j)}) = \hat{u}_\tau^* \hat{u}_\sigma^* p_\tau^{(j)} p_\sigma^{(j)}$$

From (5): $\hat{u}^*_\tau \hat{u}^*_\sigma = \bar{\delta}^*_{\tau\sigma} - \delta^*_{\tau\sigma}$

$$\therefore (\Pi_4^{(j)})^2 = (\bar{\delta}^*_{\tau\sigma} - \delta^*_{\tau\sigma})\, p_\tau^{(j)} p_\sigma^{(j)}$$
$$= \bar{\delta}^*_{\tau\sigma}\, p_\tau^{(j)} p_\sigma^{(j)} - \delta^*_{\tau\sigma}\, p_\tau^{(j)} p_\sigma^{(j)}$$

Now: $$\bar{\delta}^*_{\tau\sigma}\, p_\tau^{(j)} p_\sigma^{(j)} = (\hat{i}^*_\tau \hat{i}^*_\sigma + \hat{j}^*_\tau \hat{j}^*_\sigma + \hat{k}^*_\tau \hat{k}^*_\sigma)\, p_\tau^{(j)} p_\sigma^{(j)}$$
$$= (\Pi_1^{(j)})^2 + (\Pi_2^{(j)})^2 + (\Pi_3^{(j)})^2 = (\Pi^{(j)})^2$$

$$\delta^*_{\tau\sigma}\, p_\tau^{(j)} p_\sigma^{(j)} = p_\sigma^{(j)} p_\sigma^{(j)} = \mathring{p}_\sigma^{(j)} \mathring{p}_\sigma^{(j)} = (i\,\mathring{E}^{(j)}/c)^2$$
$$= -(m_o^{(j)}c)^2$$

$$\therefore (\Pi_4^{(j)})^2 = (\Pi^{(j)})^2 + (m_o^{(j)}c)^2$$

or, $$\Pi_4^{(j)} = E'^{(j)}/c = \left[\,(\Pi^{(j)})^2 + (m_o^{(j)}c)^2\right]^{1/2} \tag{16}$$

restricting ourselves to $\Pi_4^{(j)} > 0$

MAXWELL-BOLTZMAN DISTRIBUTION LAW

Consider a property of a system as associated with a set of k boxes with a definite value of the property attached to each box. The elements are assumed to be indistinguishable in nature and to move freely except when any kind of collision occurs between elements. One is not principally concerned with such phenomena as ionization, dissociation and radiation. Therefore the counting process of obtaining the number of discrete elements in a box is an absolute operation. Let there exist N elements as a whole and let them be arranged such that each box contains N_1, N_2, - - - - N_k

elements respectively. Then the total number of methods of arrangement is seen to be:

$$N_a = \frac{N!}{\prod_{j=1}^{k} N_j!} \tag{17}$$

with

$$\sum_{j=1}^{k} N_j = N \tag{18}$$

If the apriori probability is given by:

$g_1, g_2, - - - - - -, g_k$;

the probability associated with the above distribution will be given by:

$$G_k^N = N_a \prod_{j=1}^{k} g_j^{N_j} \tag{19}$$

with

$$\sum_{j=1}^{k} g_j = 1 \tag{20}$$

Assuming the whole system to be isolated, the components of the total momentum as well as the total energy should be conserved with respect to any reference frame. Thus the following holds:

$$\sum_j N_j P_\sigma^{(j)} = \text{a constant} \qquad (\sigma = 1, 2, 3, 4) \tag{21}$$

where $P_\sigma^{(j)}$ is the energy-momentum 4 - vector for the j-th box. From (17) and (19):

$$\log G_k^N = \log N! - \sum_j \log N_j! + \sum_j N_j \log g_j$$

For maximum probable distribution, the following relations should be satisfied simultaneously:

$$\delta \log G_k^N = 0, \quad \delta \sum_j N_j = 0, \quad \delta \sum_j N_j P_\sigma^{(j)} = 0 \tag{22}$$

Using Stirling's formula:

$$\log N! = N \log N - N + \frac{1}{2}\log 2\pi + \frac{1}{2}\log N$$
$$\simeq N \log N - N,$$

(22) can be shown to yield:

$$\sum_j \log (g_j/N_j)\,\delta N_j = 0, \quad \sum_j \delta N_j = 0, \quad \sum_j p_\sigma^{(j)}\,\delta N_j = 0 \tag{23}$$

Introducing Lagrange's undetermined multipliers:

$$\sum_j \left(\log g_j/N_j + \alpha + \alpha_\sigma p_\sigma^{(j)} \right) \delta N_j = 0 \tag{24}$$

must be a four-vector in order that the formulation be covariant and is a scalar quantity. It is seen that α_σ and α are not related to the individual elements in any specific way, but are to be related to the system as a whole. Varying δN_j arbitrarily, it is obtained:

$$\log g_j/N_j + \alpha + \alpha_\sigma p_\sigma^{(j)} = 0$$

$$N_j = g_j\, e^{\alpha + \alpha_\sigma p_\sigma^{(j)}} \tag{25}$$

By (18): $\quad \sum_j N_j = N = e^{\alpha} \sum_j g_j\, e^{\alpha_\sigma p_\sigma^{(j)}}$

$$\therefore e^{\alpha} = \left[\sum_j g_j\, e^{\alpha_\sigma p_\sigma^{(j)}} \right]^{-1} \cdot N$$

$$N_j = \frac{N g_j\, e^{\alpha_\sigma p_\sigma^{(j)}}}{\sum_j g_j\, e^{\alpha_\sigma p_\sigma^{(j)}}} \tag{26}$$

The distribution function is defined by:

$$w_j = \frac{g_j\, e^{\alpha_\sigma p_\sigma^{(j)}}}{Z} \; ; \quad \sum_j w_j = 1 \tag{27}$$

with the partition function $\quad Z = \sum_j g_j\, e^{\alpha_\sigma p_\sigma^{(j)}}$

In the case of uniform apriori probability, that is when

$$g_1 = g_2 - - - - - = g_k$$

one has

$$w_j = \frac{e^{\alpha_\sigma p_\sigma^{(j)}}}{Z} \; ; \quad Z = \sum e^{\alpha_\sigma p_\sigma^{(j)}} \tag{28}$$

(27) and (28) are the desired covariant Maxwell-Boltzman distribution functions.

PROPERTIES OF THE VECTOR α_σ

In Maxwell-Boltzman distribution law, the 4 - vector α_σ appeared to lead to a covariant formulation. To clarify the properties of this vector, one writes:

$$\alpha_\sigma p_\sigma^{(j)} = \sum_{k=1}^{3} \frac{\alpha_k m_0^{(j)} v_k^{(j)}}{\sqrt{1 - v^{(j)2}/c^2}} + i\alpha_4 \frac{m_0^{(j)} c}{\sqrt{1 - v^{(j)2}/c^2}}$$
$$= \frac{i\alpha_4 m_0^{(j)} c}{\sqrt{1 - v^{(j)2}/c^2}} \left(1 + \sum_{k=1}^{3} \frac{\alpha_k v_k^{(j)}}{i\alpha_4 c} \right) \tag{29}$$

Let

$$\alpha_4 = \frac{i\epsilon c}{\sqrt{1 - v^{*2}/c^2}} \qquad \alpha_k = \frac{\epsilon v_k^*}{\sqrt{1 - v^{*2}/c^2}}$$

$$v^{*2} = \sum_{k=1}^{3} v_k^{*2} ,$$

(k = 1, 2, 3)

where v_k^* and ϵ are four new parameters replacing α_σ (σ = 1, 2, 3, 4)

$$v_k^* = ic\frac{\alpha_k}{\alpha_4} , \qquad \epsilon = \frac{i}{c}\sqrt{\alpha_\sigma \alpha_\sigma}$$

where ϵ is a scalar invariant quantity. Now substituting (30) into (29):

$$\alpha_\sigma p_\sigma^{(j)} = m_0^{(j)} \epsilon \left[\sum_k \frac{v_k^* v_k^{(j)}}{\sqrt{1 - v^{*2}/c^2}\sqrt{1 - v^{(j)2}/c^2}} - \frac{c^2}{\sqrt{1 - v^{*2}/c^2}\sqrt{1 - v^{(j)2}/c^2}} \right]$$
$$= - m_0^{(j)} \epsilon \frac{c^2 - \sum_k v_k^* v_k^{(j)}}{\sqrt{1 - v^{*2}/c^2}\sqrt{1 - v^{(j)2}/c^2}} \tag{31}$$

Now define the velocity 4 - vector for a particle at rest in the secondary frame by (15):

$$u_\sigma^\circ = (0, \quad 0, \quad 0, \quad ic)$$

which leads to:

$$u_\sigma^* = L_{\sigma\tau}^* u_\tau^\circ = L_{\sigma 4}^* u_4^\circ = ic\, L_{\sigma 4}^* \tag{32}$$

Comparing with (4), one has:

$$u_\sigma^{*} = c\,\hat{u}_\sigma^{*}$$

Similary one defines:

$$u_\sigma^{(j)} = L_{\sigma 2}^{(j)}\,u_2^{0} = ic\,L_{\sigma 4}^{(j)} \tag{33}$$

where

$$u_\sigma^{0} = (0,\ 0,\ 0,\ ic)$$

is the velocity 4 - vector for a particle at rest in the rest frames.

Thus:

$$u_\sigma^{*} u_\sigma^{(j)} = -c^2\,L_{\sigma 4}^{*}\,L_{\sigma 4}^{(j)}$$

$$= -\frac{c^2 - \sum_k v_k^{*} v_k^{(j)}}{\sqrt{1 - v^{*2}/c^2}\,\sqrt{1 - v^{(j)2}/c^2}}$$

Substituting the above into (31):

$$\alpha_\sigma p_\sigma^{(j)} = m_0^{(j)}\,\epsilon\,u_\sigma^{*}\,u_\sigma^{(j)} = m_0^{(j)}\,\epsilon\,u_\sigma^{\Delta}\,u_\sigma'^{(j)}$$

$$= -\frac{\epsilon\,m_0^{(j)} c^2}{\sqrt{1 - v'^{(j)2}/c^2}} = -\epsilon\,E'^{(j)}$$

where $u_\sigma'^{(j)}$ is the velocity 4 - vector of the j-th particle as measured by the secondary observer.

Thus the distribution law (28) is reduced to a form identical with the classical Maxwell-Boltzman law, namely:

$$w_j = \frac{e^{-\epsilon E'^{(j)}}}{\sum_j e^{-\epsilon E'^{(j)}}}\,, \quad E'^{(j)} = c\,\pi_4^{(j)} = c\left[(\pi^{(j)})^2 + (m_0^{(j)}c)^2\right]^{1/2} \tag{34}$$

Thus finally one sets

$$\alpha_\sigma p_\sigma^{(j)} = m_0^{(j)}\,\epsilon\,u_\sigma^{*}\,u_\sigma^{(j)} = \epsilon\,u_\sigma^{*}\,p_\sigma^{(j)}$$

$$\therefore\ \alpha_\sigma = \epsilon\,u_\sigma^{*} \tag{35}$$

Corresponding further with the classical formula, one may obviously set

$$\epsilon = [kT]^{-1} \tag{36}$$

where K is the Boltzman constant and T is the scalar invariant temperature as measured at rest in the secondary frame. (36) can be further assured by considering the entropy of a system as a world invariant quantity defined by:

$$S = -Nk \sum_j w_j \log w_j \tag{37}$$

It can be easily verified that the above definition gives the statistical interpretation of the tendency of increasing entropy for which results from mixing two systems (10).

From (28) with (35):

$$w_j = Z^{-1} e^{\epsilon u_\sigma^* p_\sigma^{(j)}} \quad , \quad Z = \sum_j e^{\epsilon u_\sigma^* p_\sigma^{(j)}}$$

$$\log w_j = \epsilon u_\sigma^* p_\sigma^{(j)} - \log Z$$

$$\sum_j w_j \log w_j = \epsilon u_\sigma^* \sum_j w_j p_\sigma^{(j)} - \log Z \sum_j w_j$$

Since, $\sum_j w_j = 1 , \quad \sum w_j p_\sigma^{(j)} = \overline{p_\sigma}$ (average)

$$\sum w_j \log w_j = \epsilon u_\sigma^* \overline{p_\sigma} - \log Z \tag{38}$$

Substituting this into (37), one obtains:

$$S = -Nk \left(\epsilon u_\sigma^* \overline{p_\sigma} - \log Z \right) \tag{39}$$

Now,

$$\frac{\partial \log Z}{\partial \epsilon} = \frac{1}{Z} \frac{\partial Z}{\partial \epsilon} = \frac{1}{Z} \frac{\partial}{\partial \epsilon} \left(\sum_j e^{\epsilon u_\sigma^* p_\sigma^{(j)}} \right)$$

$$= \frac{1}{Z} \sum_j \left(u_\sigma^* p_\sigma^{(j)} e^{\epsilon u_\sigma^* p_\sigma^{(j)}} \right) = \sum_j u_\sigma^* p_\sigma^{(j)} w_j \tag{40}$$

$$= u_\sigma^* \overline{p_\sigma} = \overline{E}'$$

$$\therefore \overline{E}' = u_\sigma^* \overline{p_\sigma} = \frac{\partial \log Z}{\partial \epsilon}$$

$E'^{(j)} = u_\sigma^* p_\sigma^{(j)}$ depends in general on an external parameter tensor $\lambda^{\alpha\beta\cdots}_{\sigma\rho\cdots}$, which is to include all physically interesting variables in a system and can be considered as independent of ϵ. Then the thermodynamic functions will depend not only on ϵ but also on $\lambda^{\alpha\beta\cdots}_{\sigma\rho\cdots}$. The force acting in the direction of $\lambda^{\alpha\beta\cdots}_{\sigma\rho\cdots}$ is now given by:

$$F^{\sigma\rho\cdots(j)}_{\alpha\beta\cdots} = \frac{\partial(u_\sigma^* p_\sigma^{(j)})}{\partial\lambda^{\alpha\beta\cdots}_{\sigma\rho\cdots}}$$

with its average value:

$$\overline{F^{\sigma\rho\cdots}_{\alpha\beta\cdots}} = \sum_j w_j F^{\sigma\rho\cdots(j)}_{\alpha\beta\cdots} \tag{41}$$

Now:

$$\frac{\partial \log Z}{\partial\lambda^{\alpha\beta\cdots}_{\sigma\rho\cdots}} = \frac{1}{Z}\frac{\partial Z}{\partial\lambda^{\alpha\beta\cdots}_{\sigma\rho\cdots}} = \frac{1}{Z}\frac{\partial}{\partial\lambda^{\alpha\beta\cdots}_{\sigma\rho\cdots}}\left(\sum_j e^{\epsilon u_\sigma^* p_\sigma^{(j)}}\right)$$

$$\begin{aligned} &= \frac{\epsilon}{Z}\sum_j e^{\epsilon u_\sigma^* p_\sigma^{(j)}}\frac{\partial(u_\sigma^* p_\sigma^{(j)})}{\partial\lambda^{\alpha\beta\cdots}_{\sigma\rho\cdots}} \\ &= \epsilon\sum_j w_j F^{\sigma\rho\cdots(j)}_{\alpha\beta\cdots} = \epsilon\,\overline{F^{\sigma\rho\cdots}_{\alpha\beta\cdots}} \end{aligned} \tag{42}$$

Now let $\lambda^{\alpha\beta\cdots}_{\sigma\rho\cdots}$ be changed by a small amount $\delta\lambda^{\alpha\beta\cdots}_{\sigma\rho\cdots}$ and ϵ by $\delta\epsilon$. This will change the canonical ensemble into another neighboring but still canonical one, with slightly different entropy. Then the average work, as a scalar quantity, done for an element in this process can be given by

$$\delta\bar{w} = \overline{F^{\sigma\rho\cdots}_{\alpha\beta\cdots}}\,\delta\lambda^{\alpha\beta\cdots}_{\sigma\rho\cdots} \tag{43}$$

Then by (42):

$$\begin{aligned} \epsilon\delta\bar{w} &= \epsilon\,\overline{F^{\sigma\rho\cdots}_{\alpha\beta\cdots}}\,\delta\lambda^{\alpha\beta\cdots}_{\sigma\rho\cdots} \\ &= \frac{\partial\log Z}{\partial\lambda^{\alpha\beta\cdots}_{\sigma\rho\cdots}}\,\delta\lambda^{\alpha\beta\cdots}_{\sigma\rho\cdots} \end{aligned} \tag{44}$$

Since the partition function Z now depends on both ϵ and $\lambda^{\alpha\beta\cdots}_{\sigma\rho\cdots}$:

$$\delta(\log Z) = \frac{\partial\log Z}{\partial\epsilon}\delta\epsilon + \frac{\partial\log Z}{\partial\lambda^{\alpha\beta\cdots}_{\sigma\rho\cdots}}\delta\lambda^{\alpha\beta\cdots}_{\sigma\rho\cdots}$$

Substituting this into (44):

$$\Theta\delta\bar{w} = \delta \log Z - \frac{\partial \log Z}{\partial \Theta}\delta\Theta$$

Further substituting (40) into the above equation:

$$\Theta\delta\bar{w} = \delta \log Z - (u_\sigma^{\neq}\bar{p}_\sigma)\delta\Theta \tag{45}$$

Using (45):

$$\begin{aligned} k\Theta\left\{N\delta(u_\sigma^{\neq}\bar{p}_\sigma) - N\delta\bar{w}\right\} &= Nk\left\{\Theta\delta(u_\sigma^{\neq}\bar{p}_\sigma) - \Theta\delta\bar{w}\right\} \\ &= Nk\left\{\Theta\delta(u_\sigma^{\neq}\bar{p}_\sigma) - \delta\log Z + (u_\sigma^{\neq}\bar{p}_\sigma)\delta\Theta\right\} \\ &= Nk\left\{\delta(\Theta u_\sigma^{\neq}\bar{p}_\sigma) - \delta\log Z\right\} \\ &= \delta\left\{Nk\left(\Theta u_\sigma^{\neq}\bar{p}_\sigma - \log Z\right)\right\} \end{aligned}$$

From (39), thus

$$\delta S = -k\Theta\left\{N\delta(u_\sigma^{\neq}\bar{p}_\sigma) - N\delta\bar{w}\right\}$$

Now:

$$N\delta\bar{w} = dW$$

(total work done by the system)

$$N\delta(u_\sigma^{\neq}\bar{p}_\sigma) = N\delta(-\bar{E}') = -N\delta\bar{E}' = -\delta U$$

(δU is total internal energy change)

$$\delta S = k\Theta(\delta U + \delta W)$$

If the total heat absorbed by the system is δQ, one has:

$$\delta Q = \delta U + \delta W$$

thus yielding:

$$\delta S = k\Theta\delta Q \tag{46}$$

Thus as it was given in (36), it is again found that:

$$\Theta = [kT]^{*}$$

Thus finally by (35) and (36):

$$\alpha_\sigma = \Theta u_\sigma^{\neq} = u_\sigma^{\neq}[kT]^{-1} \tag{47}$$

(47) is seen to be in agreement with the previous statement that $d\sigma$ is only related to a system as a whole and not to an individual component system in any specific way. With (47), (27), and (28) one has:

$$W_j = g_j e^{\epsilon u_\sigma^* p_\sigma^{(j)}} \cdot Z^{-1} \quad ; \quad Z = \sum_j g_j e^{\epsilon u_\sigma^* p_\sigma^{(j)}} \tag{48}$$

and

$$w_j = e^{\epsilon u_\sigma^* p_\sigma^{(j)}} \cdot Z^{-1} \quad ; \quad Z = \sum_j e^{\epsilon u_\sigma^* p_\sigma^{(j)}} \tag{49}$$

For a given physical quantity, for instance $A_\sigma^{(j)}$, its average value will be given by using (49).

$$\begin{aligned}
\overline{A}_\sigma &= \sum_j A_\sigma^{(j)} w_j = \sum_j A_\sigma^{(j)} e^{\epsilon u_\sigma^* p_\sigma^{(j)}} \cdot Z^{-1} \\
&= \sum_j \perp_{\sigma z}^{*} A_z^{\prime(j)} e^{\epsilon u_\sigma^* p_\sigma^{\prime(j)}} \cdot Z'^{-1} \quad ; \quad Z' = Z = \sum_j e^{\epsilon u_\sigma^* p_\sigma^{\prime(j)}} \\
&= \perp_{\sigma z}^{*} \sum_j A_z^{\prime(j)} w_j' \qquad ; \quad w_j' = w_j = \frac{e^{\epsilon u_\sigma^* p_\sigma^{\prime(j)}}}{Z'} \\
&= \perp_{\sigma z}^{*} \overline{A}_z'
\end{aligned}$$

where $\overline{A}_z'$ is the average value taken relative to the secondary frame and will be given by:

$$\overline{A}_z' = \sum_j A_z^{\prime(j)} w_j' = \sum_j A_z^{\prime(j)} \frac{e^{\epsilon u_\sigma^* p_\sigma^{\prime(j)}}}{\sum_j e^{\epsilon u_\sigma^* p_\sigma^{\prime(j)}}}$$

Since

$$\epsilon u_\sigma^* p_\sigma^{\prime(j)} = -E^{\prime(j)}/kT \quad , \quad \Pi_4^{(j)} = E^{\prime(j)}/c = \left[(\Pi^{(j)})^2 + (m_0^{(j)} c)^2 \right]^{1/2}$$

$$\overline{A}_z' = \frac{\sum A_z^{\prime(j)} e^{-E^{\prime(j)}/kT}}{\sum_j e^{-E^{\prime(j)}/kT}} = \frac{\sum_j A_z^{\prime(j)} e^{-c\left[\Pi^{(j)2} + (m_0^{(j)}c)^2\right]^{1/2}/kT}}{\sum_j e^{-c\left[\Pi^{(j)2} + (m_0^{(j)}c)^2\right]^{1/2}/kT}} \tag{50}$$

For a system consisting of an ideal gas, the 6 - dimensional phase space (ξ_k, Π_k) will be introduced, and it will be divided into cells of size h^3 so that the number of cells per a phase volume $d\phi = \prod_{k=1}^{3} d\xi_k d\Pi_k$, which itself

is invariant according to (19) and (20), is:

$$\frac{d\phi}{h^3} = \frac{\prod_{k=1}^{3} d\xi_k \, d\pi_k}{h^3}$$

Then the average value of any function of ξ_k and π_k, $f(\xi_k, \pi_k)$ can be obtained by:

$$\overline{f(\xi_k, \pi_k)} = \frac{\int \cdots \int f(\xi_k, \pi_k)\, e^{-c[\pi^2 + (m_0 c)^2]^{1/2}/kT} \prod_{k=1}^{3} \frac{d\xi_k d\pi_k}{h^3}}{\int \cdots \int e^{-c[\pi^2 + (m_0 c)^2]^{1/2}/kT} \prod_{k=1}^{3} \frac{d\xi_k d\pi_k}{h^3}}$$

$$= \frac{\int \cdots \int f(\xi_k, \pi_k)\, e^{-c[\pi^2 + (m_0 c)^2]^{1/2}/kT} \prod_{k=1}^{3} d\xi_k d\pi_k}{\int \cdots \int e^{-c[\pi^2 + (m_0 c)^2]^{1/2}/kT} \prod_{k=1}^{3} d\xi_k d\pi_k}$$

In particular we get $\overline{\pi_k} = 0$, if the limits of integration are, in approximation, $\pm\infty$. Unfortunately still no general method for the evaluation of (51) for $m_0 \neq 0$ seems to be available (11).

EQUATIONS OF MOTION

Now a formulation of the Gibb's relativistic covariant statistical mechanics will be discussed. Consider a system of particles whose relative position 4 - vector $x_\sigma^{(j)}$ satisfies the following operations, namely:

$$-\hat{u}_\sigma^{*} x_\sigma^{(j)} = -\hat{u}_\sigma^{\circ} x_\sigma'^{(j)} = ct' \qquad (j = 1, 2, 3, - - - - -) \tag{52}$$

where t' is an invariant time parameter equal to the relative time as read by the synchronized clocks in the secondary frame. Since $\xi_k^{(j)}$ (k = 1, 2, 3) defined in (13) is also scalar invariant to the Lorentz transformation, any scalar function of these quantities, ξ_k & t', is also invariant. Indeed a new

generalized Lagrangian function L for a system of particles can be defined as a function of $\xi_K^{(j)}$ and $\dot{\xi}_K^{(j)} = \frac{d\xi_K^{(j)}}{dt'}$ with t' as parameter. The covariant formulation of Hamilton principle can then be given by

$$\delta I = \delta \int_{t_1'}^{t_2'} L\left(\xi_K^{(j)}, \dot{\xi}_K^{(j)}, t'\right) dt' = 0 \tag{53}$$

It is not always possible to set up a completely covariant formulation of a given problem in this manner. The explicit form of the potentials are determined by the nature of the forces involved, and not all types of forces are available in a covariant form. Action-at-a-distance forces cannot be involved in this treatment and thus one has only to consider those theories from which the concept of action-at-a-distance can be eliminated. This is possible in the theory of collisions and one shall be concerned with an ideal gas system in which the molecules are assumed to be infinitesimal in size and for which the interaction takes place only at the instant that the distance between two particles is negligible. Before and after this infinitesimally short time interval of collision, the motion of the particles is not accelerated and during the short period of interaction, the conservation laws of momentum and energy are fulfilled.

The Hamilton principle (52) will lead to the Lagrangian equations of motion, namely:

$$\frac{d}{dt'}\left(\frac{\partial L}{\partial \dot{\xi}_K^{(j)}}\right) - \frac{\partial L}{\partial \xi_K^{(j)}} = 0 \tag{54}$$

These describe the motion of the particles relative to the secondary frame as seen by the primary frame.

The canonically conjugate momentum of $\xi_K^{(j)}$ is defined by,

$$\frac{\partial L}{\partial \dot{\xi}_K^{(j)}} = \Pi_K^{(j)} \tag{55}$$

From (16), (20):

$$\Pi_k^{(j)} = \hat{e}^*_{k\sigma} p_\sigma^{(j)} = \hat{e}'_{k\sigma} p_\sigma'^{(j)} = p_k'^{(j)}$$
$$= m_0^{(j)} u_k'^{(j)} = \frac{m_0^{(j)} \dot{\xi}_k^{(j)}}{\sqrt{1 - \dot{\xi}^{(j)2}/c^2}} \quad (k = 1, 2, 3) \tag{56}$$

$$\dot{\xi}^{(j)2} = \dot{\xi}_1^{(j)2} + \dot{\xi}_2^{(j)2} + \dot{\xi}_3^{(j)2}, \quad \hat{e}^*_{k\sigma} = (\hat{i}^*_\sigma, \hat{j}^*_\sigma, \hat{k}^*_\sigma) \tag{57}$$

$$\frac{\partial L}{\partial \dot{\xi}_k^{(j)}} = \Pi_k^{(j)} = \frac{m_0^{(j)} \dot{\xi}_k^{(j)}}{\sqrt{1 - \dot{\xi}^{(j)2}/c^2}}$$

This can be satisfied by setting:

$$L = \sum_j L^{(j)} + L'$$

where $L^{(j)} = -m_0^{(j)} c^2 \sqrt{1 - \dot{\xi}^{(j)2}/c^2}$ and L' is some function of $\xi_k^{(j)}$ only.

Thus:

$$L = \sum_j \left(-m_0^{(j)} c^2 \sqrt{1 - \dot{\xi}^{(j)2}/c^2} \right) + L' \tag{58}$$

From (56):

$$\sum_j \Pi_k^{(j)} = \sum_j \hat{e}^*_{k\lambda} p_\lambda^{(j)} = \hat{e}^*_{k\lambda} \sum_j p_\lambda^{(j)}$$
$$(k = 1, 2, 3)$$

According to the principle of conservation of energy and momentum:

$$\sum_j p_\lambda^{(j)} = \text{a constant} \quad (\ \ = 1, 2, 3, 4)$$

Also,

$$\hat{e}^*_{k\lambda} = \text{a constant} \quad \begin{matrix}(k = 1, 2, 3) \\ (\ \ = 1, 2, 3, 4)\end{matrix}$$

Therefore:

$$\sum_j \Pi_k^{(j)} = \text{a constant} \quad (k = 1, 2, 3) \tag{59}$$

Summing up (54) over j:

$$\frac{d}{dt}\left(\sum_j \frac{\partial L}{\partial \dot{\xi}_k^{(j)}}\right) - \sum_j \frac{\partial L}{\partial \xi_k^{(j)}} = 0$$

By (55):

$$\frac{d}{dt'}\left(\sum_j \Pi_k^{(j)}\right) - \sum_j \frac{\partial L}{\partial \xi_k^{(j)}} = 0$$

By (59):

$$\frac{d}{dt'}\left(\sum_j \Pi_k^{(j)}\right) = 0$$

$$\sum_j \frac{\partial L}{\partial \xi_k^{(j)}} = 0 \qquad (k = 1, 2, 3) \qquad (60)$$

As $L = \sum_j L^{(j)} + L'$ and only L' involves $\xi_k^{(j)}$, this yields:

$$\sum_j \frac{\partial L'}{\partial \xi_k^{(j)}} = 0 \qquad (k = 1, 2, 3) \qquad (60)'$$

(60)' is a homogeneous linear partial differential equation of the first order with its coefficients all unity. Its subsidiary equations may be written as: $d\xi_k^{(1)} = d\xi_k^{(2)} = - - - = d\xi_k^{(j)} = - - - - -$, $d L' = 0$, $(k = 1, 2, 3)$.

(n-1) independent integrals of the above can be obtained, namely:

$$b_k^{(j,1)} = \xi_k^{(j)} - \xi_k^{(1)} = \text{a constant.}$$

$$b_k^{(j,2)} = \xi_k^{(j)} - \xi_k^{(2)} = \text{a constant.}$$

- - - - - - - - - -

$$b_k^{(j,i)} = \xi_k^{(j)} - \xi_k^{(i)} = \text{a constant.}$$

- - - - - - - - - -

$(i \neq j;\ k = 1, 2, 3,;\ i, j = 1, 2 - - - - -)$
$(k = 1, 2, 3.)$

Therefore the general integral of (60)' is given by:

$$L' = -V\left(b_k^{(i,j)}\right) = -V\left(\xi_k^{(j)} - \xi_k^{(i)}\right)$$

$(i \neq j;\ k = 1, 2, 3,;\ i, j = 1, 2 - - - - -)$

Substituting the above into (58):

$$L = \sum_j \left(-m_0^{(j)} c^2 \sqrt{1 - \dot{\xi}^{(j)2}/c^2}\right) - V\left(\xi_k^{(j)} - \xi_k^{(i)}\right) \qquad (61)$$

Further the Hamiltonian function is defined by:

$$H = \sum_{jk} \pi_k^{(j)} \dot{\xi}_k^{(j)} - L$$

$$= \sum_{jk} \pi_k^{(j)} \dot{\xi}_k^{(j)} + \sum_j m_0^{(j)} c^2 \sqrt{1 - \dot{\xi}^{(j)2}/c^2} + V$$

Substituting (56):

$$H = \sum_{jk} \frac{m_0^{(j)} \dot{\xi}_k^{(j)}}{\sqrt{1 - \dot{\xi}^{(j)2}/c^2}} + \sum_j m_0^{(j)} c^2 \sqrt{1 - \dot{\xi}^{(j)2}/c^2} + V$$

$$= \sum_j \frac{m_0^{(j)} c^2}{\sqrt{1 - \dot{\xi}^{(j)2}/c^2}} + V$$

$$\therefore H = \sum_j \frac{m_0^{(j)} c^2}{\sqrt{1 - \dot{\xi}^{(j)2}/c^2}} + V(\xi_k^{(j)} - \xi_k^{(i)}) \tag{62}$$

The relations appearing in the formulation are all alike in form with the classical formulation. But in essence the two cases are different. In this formulation, only invariant quantities are involved and thus the formulation is completely covariant, but the classical formulation is not. For example:

$$\dot{\xi}_k^{(j)} = \frac{d\xi_k^{(j)}}{dt'} = \frac{\hat{e}^{*}_{k)\lambda} u_\lambda^{(j)}}{\widetilde{u}_\sigma \hat{u}_\sigma^{(j)}}$$

and thus:

$$H = \sum_j \frac{m_0^{(j)} c^2}{\left\{ 1 - \sum_{k=1}^{3} \left(\frac{\hat{e}^{*}_{k)\lambda} u_\lambda^{(j)}}{\widetilde{u}_\sigma \hat{u}_\sigma^{(j)}} \right)^2 / c^2 \right\}^{1/2}} + V(\hat{e}^{*}_{k)\sigma} x_\sigma^{(j)} - \hat{e}^{*}_{k)\sigma} x_\sigma^{(i)})$$

It is seen that V is solely dependent on the relative distances of the particles, and thus, in the ideal gas problem, V shall be appreciable instantaneously at the limit of $\xi_k^{(j)} \to \xi_k^{(i)}$ (k = 1, 2, 3) etc., and otherwise vanishes. V is only to control such instantaneous collision process

previously described, and one is thus assuming that its interacting intervals for collision are so short that the only sum $\sum_{j} \frac{m_0^{(j)}c^2}{\sqrt{1-\xi^{(j)2}/c^2}}$, the total relativistic kinetic energy, itself can, at any instant, express the total energy of the system. One will not be concerned about the explicit form of V, but V will be kept in the Hamiltonian H only to indicate the fact that the particles are not completely free but execute collisions between particles, changing from one free motion state to another. Therefore, for any given instant, one may put:

Total energy of the system (relative to the secondary frame) $= \sum_{j} \frac{m_0^{(j)}c^2}{\sqrt{1-\xi^{(j)2}/c^2}} = \sum_{j}\left(-\hat{u}_\sigma^{*} p_\sigma^{(j)}\right)$

$$= -\hat{u}_\sigma^{*}\left(\sum_{j} p_\sigma^{(j)}\right)$$

where $P_\sigma^{(j)}$ is the energy-momentum 4 - vector for j-th particle as defined by (16).

Finally Hamilton equations of motion can be given by:

$$\frac{\partial H}{\partial \Pi_k^{(j)}} = \frac{d\xi_k^{(j)}}{dt'} \quad , \quad \frac{\partial H}{\partial \xi_k^{(j)}} = -\frac{d\Pi_k^{(j)}}{dt'} \tag{63}$$

$$(k = 1, 2, 3)$$

STATIONARY STATE

A phase space is defined by introducing position-momentum coordinates ($\xi_k^{(j)}, \Pi_k^{(j)}$) into a single coordinate organization for an ensemble. Introducing the phase density ρ which is a world invariant quantity, the phase currents along the phase coordinate axes can be defined by:

$$D = \left(D\xi_k^{(j)},\ D\pi_k^{(j)}\right) \tag{64}$$

where $$D_{\xi_k^{(j)}} = \rho \frac{d\xi_k^{(j)}}{dt'} , \quad D_{\pi_k^{(j)}} = \rho \frac{d\pi_k^{(j)}}{dt'} \tag{65}$$

Since the number of representative points in the ensemble is conserved, the equation of continuity shall be satisfied:

$$\frac{\partial \rho}{\partial t'} + \sum_{j,k} \left(\frac{\partial D_{\xi_k^{(j)}}}{\partial \xi_k^{(j)}} + \frac{\partial D_{\pi_k^{(j)}}}{\partial \pi_k^{(j)}} \right) = 0 \tag{66}$$

With the use of the Hamilton equations of motion:

$$\frac{\partial \rho}{\partial t'} + \sum_{j,k} \left[\frac{\partial(\rho\, \dot{\xi}_k^{(j)})}{\partial \xi_k^{(j)}} + \frac{\partial(\rho\, \dot{\pi}_k^{(j)})}{\partial \pi_k^{(j)}} \right] \quad ; \quad \dot{\pi}_k^{(j)} = \frac{d\pi_k^{(j)}}{dt'}$$

$$= \frac{\partial \rho}{\partial t'} + \sum_{j,k} \left[\frac{\partial \rho}{\partial \xi_k^{(j)}} \frac{d\xi_k^{(j)}}{dt'} + \frac{\partial \rho}{\partial \pi_k^{(j)}} \frac{d\pi_k^{(j)}}{dt'} \right]$$

$$= \frac{d\rho}{dt'} = 0$$

where use is made of the fact that ρ is to be a function of the form:

$$\rho = \rho\ (\xi_k^{(j)}, \pi_k^{(j)}, t')$$

and the differentistions of H with respect to $\xi_k^{(j)}$ and $\pi_k^{(j)}$ are interchangeable. Thus covariant Liouville's theorem is given by:

$$\frac{\partial \rho}{\partial t'} = 0 \tag{67}$$

or, by using the Poisson's brackets, it can also be written as:

$$\frac{\partial \rho}{\partial t'} + \sum_{j,k} \left(\frac{\partial \rho}{\partial \xi_k^{(j)}} \frac{\partial H}{\partial \pi_k^{(j)}} - \frac{\partial \rho}{\partial \pi_k^{(j)}} \frac{\partial H}{\partial \xi_k^{(j)}} \right)$$

$$= \frac{\partial \rho}{\partial t'} + [\rho, H] = 0$$

$$\frac{\partial \rho}{\partial t'} + [\rho, H] = 0 \tag{68}$$

The stationary condition will be given by:

$$\frac{\partial \rho}{\partial t'} = 0 \tag{69}$$

to yield:

$$[\rho, H] = 0 \tag{70}$$

The condition for a steady state is satisfied, therefore, by choosing the phase density function ρ to be a function of the constants of the motion of the system, for then the Poisson bracket with H must vanish. In (59), it is shown that $\sum_j \pi_k^{(j)}$ (k = 1, 2, 3) are conserved in an isolated system. Further by (16), it is also seen that $\sum_j \pi_4^{(j)}$ is also conserved. They are the components of total momentum and energy of a system as measured relative to the secondary frame. Since these quantities are the most significant conserved quantities under consideration, one may indeed assume ρ to be a function of total components of such quantities. It should, however, be noted that angular momentum and isotopic spin, etc. are also conserved in general, but this treatment has previously been restricted to the above simpler case.

Thus the phase density function can be given by:

$$\rho = \rho\,(\pi_1, \pi_2, \pi_3, \pi_4) \tag{71}$$

where

$$\sum_j \pi_\sigma^{(j)} = \pi_\sigma \qquad (\sigma = 1, 2, 3, 4)$$

CANONICAL DISTRIBUTION

The volume element in the phase space can be given by:

$$d\phi = \prod_{j=1}^{n} \prod_{k=1}^{3} d\xi_k^{(j)}\, d\pi_k^{(j)} \tag{72}$$

if there exist n elements in an ensemble. The probability coefficient W

in the phase space is defined by:

$$W = \rho / n \ , \quad \int W \, d\phi = 1 \tag{73}$$

For a physical quantity A characteristic of the system being described, the average value is given by:

$$\bar{A} = \int A \, W \, d\phi \tag{74}$$

where the integration is to be taken over the whole phase occupied by the ensemble.

For the ideal gas problem, an ensemble is resolved into a number of component systems which are independent of each other save for the possibility of energy-momentum exchange. Thus in an ensemble of the aggregation of the gas molecules of a given mass, the component system might be an individual molecule and its four energy-momentum components may be exchanged during collision process between particles. Denote the resultant system by S and its n-component system by $s_1, s_2, - - -, s_n$. Since each component system has three degrees of freedom, a six-dimensional phase space ($\xi_1^{(j)}, \xi_2^{(j)}, \xi_3^{(j)}, \pi_1^{(j)}, \pi_2^{(j)}, \pi_3^{(j)}$) for each component system is introduced. The total number of degrees of freedom of S is 3 n and, for this resultant system, 6 n-dimensional Γ space will be employed. 6-dimensional μ_j-space are employed for the component systems.

Now by the previous argument, the phase probability coefficient for the resultant system is a function of the total components of energy and momentum relative to the secondary frame, that is, $W(\pi_1, \pi_2, \pi_3, \pi_4)$.

To be specific, consider an ensemble for a single particle system. For instance, one may consider the case where all the particles are taken out of the system except the j-th particle. Then one has:

$$W = W(\pi_1^{(j)}, \pi_2^{(j)}, \pi_3^{(j)}, \pi_4^{(j)})$$

It must be emphasized that in constituting the canonical ensemble for a single particle, one cannot assume that the energy and momentum of the canonical ensemble are fixed. By the very nature of the canonical ensemble, it must be assumed that the energy and momentum of the single particle system being described is able to fluctuate. It cannot, therefore, be considered free in the usual dynamical sense, even though its energy and momentum are treated as wholly kinetic.

Then returning to the original ensemble, it is further assumed that there will presumably exist a phase probability:

$$W_j = W_j\,(\pi_1^{(j)}, \pi_2^{(j)}, \pi_3^{(j)}, \pi_4^{(j)}) \tag{75}$$

in each μ_j-space, which is a function of the instantaneous values of energy and momentum of the component system.

Now the probability that the phase point for system s_1 lies in $d\phi_1 = \prod_{k=1}^{3} d\xi_k^{(1)}\, d\pi_k^{(1)}$ of μ_1-space, - - - -, for system s_j lies in $d\phi_j = \prod_{k=1}^{3} d\xi_k^{(j)}\, d\pi_k^{(j)}$ of μ_j-space, - - - - -, etc., will be given by:

$$\prod_{j=1}^{n} w_j\, d\phi_j$$

Since the component systems are assumed to be independent, the probability will be given by:

$$\prod_{j=1}^{n} w_j\, d\phi_j = w\, d\phi$$

As $\prod_{j=1}^{n} d\phi_j = d\phi$, it yields:

$$\prod_{j=1}^{n} w_j(\pi_1^{(j)}, \pi_2^{(j)}, \pi_3^{(j)}, \pi_4^{(j)}) = w(\pi_1, \pi_2, \pi_3, \pi_4) \tag{76}$$

Since:

$$\frac{\partial w}{\partial \pi_\sigma^{(j)}} = \frac{\partial w}{\partial \pi_\sigma}\frac{\partial \pi_\sigma}{\partial \pi_\sigma^{(j)}} = \frac{\partial w}{\partial \pi_\sigma}$$

One can set:

$$\frac{\partial W}{\partial \Pi_\sigma^{(j)}} = \frac{\partial W}{\partial \Pi_\sigma^{(i)}} = \frac{\partial W}{\partial \Pi_\sigma} \qquad (i, j = 1, 2, ---, n) \tag{77}$$

Also:

$$\frac{\partial W}{\partial \Pi_\sigma^{(j)}} = \frac{\partial(\prod_j w_j)}{\partial \Pi_\sigma^{(j)}} = \frac{\partial w_j}{\partial \Pi_\sigma^{(j)}} \prod_{k \neq j} w_k$$

Substituting into (77):

$$\frac{\partial w_j}{\partial \Pi_\sigma^{(j)}} \prod_{k \neq j} w_k = \frac{\partial w_i}{\partial \Pi_\sigma^{(i)}} \prod_{k \neq i} w_k = \frac{\partial W}{\partial \Pi_\sigma}$$

Dividing by $W = \prod_k w_k$:

$$\alpha_\sigma' = \frac{1}{w_j} \frac{\partial w_j}{\partial \Pi_\sigma^{(j)}} = \frac{1}{w_i} \frac{\partial w_i}{\partial \Pi_\sigma^{(i)}} = \frac{1}{W} \frac{\partial W}{\partial \Pi_\sigma} \tag{78}$$

$$(i, j = 1, 2, -----)$$

or

$$\begin{aligned}
\alpha_\sigma' &= \frac{1}{w_1(\Pi_1^{(1)}, --, \Pi_4^{(1)})} \frac{\partial w_1(\Pi_1^{(1)}, --, \Pi_4^{(1)})}{\partial \Pi_\sigma^{(1)}} \\
&= \frac{1}{w_2(\Pi_1^{(2)}, ---, \Pi_4^{(2)})} \frac{\partial w_2(\Pi_1^{(2)}, ---, \Pi_4^{(2)})}{\partial \Pi_\sigma^{(2)}} \\
&= ---------------------- \\
&= \frac{1}{w_n(\Pi_1^{(n)}, ---, \Pi_4^{(n)})} \frac{\partial w_n(\Pi_1^{(n)}, --, \Pi_4^{(n)})}{\partial \Pi_\sigma^{(n)}} \\
&= \frac{1}{W(\Pi_1, ---, \Pi_4)} \frac{\partial W(\Pi_1, ---, \Pi_4)}{\partial \Pi_\sigma} .
\end{aligned}$$

Thus it is seen that the variables are completely separated and thus α_σ'

is a constant. Further by the last term, it is also seen that α_σ' is related to the system as a whole and not to any individual component system in any specific way, as was α_σ in the Maxwell-Boltzman distribution law. Equation (78) can be written now:

$$\frac{1}{w_j}\,\frac{\partial w_j}{\partial \Pi_1^{(j)}} = \alpha_1' \quad \text{(a constant)}$$

$$\frac{1}{w_j}\,\frac{\partial w_j}{\partial \Pi_2^{(j)}} = \alpha_2' \quad \text{(a constant)}$$

$$(j = 1,\ 2,\ ---,\ n)$$

$$\frac{1}{w_j}\,\frac{\partial w_j}{\partial \Pi_3^{(j)}} = \alpha_3' \quad \text{(a constant)}$$

$$\frac{1}{w_j}\,\frac{\partial w_j}{\partial \Pi_4^{(j)}} = \alpha_4' \quad \text{(a constant)} \qquad (79)$$

For (79), a possible solution is:

$$\log W_j = \alpha_\sigma' \Pi_\sigma^{(j)} + \beta^{(j)}$$

where, in the right hand side, the summation convention notation is used. However, it should be noticed that α_σ', in this new formulation, is to be an invariant quantity in contrast with α_σ of the Maxwell-Boltzman distribution law which was four vector.

$$\therefore W_j = e^{\beta^{(j)}} e^{\alpha_\sigma' \Pi_\sigma^{(j)}}$$

$$\therefore W = \prod_j w_j = e^{B} e^{\sum_j \alpha_\sigma' \Pi_\sigma^{(j)}} = e^{B} e^{\alpha_\sigma' \sum_j \Pi_\sigma^{(j)}}$$

$$= e^{B} e^{\alpha_\sigma' \Pi_\sigma}$$

where $\quad B = \sum_j \beta^{(j)}$.

By (73):

$$\int w\, d\phi = e^{B} \int e^{\alpha_\sigma' \Pi_\sigma}\, d\phi = 1$$

$$\therefore e^{B} = \left[\int e^{\alpha_\sigma' \Pi_\sigma} d\phi \right]^{-1}$$

$$w = \frac{e^{\alpha_\sigma' \Pi_\sigma}}{\int e^{\alpha_\sigma' \Pi_\sigma} d\phi} \tag{80}$$

the partition function is,

$$Z = e^{-B} = \int e^{\alpha_\sigma' \Pi_\sigma} d\phi$$

Now, it can be shown that, with $p_\sigma = \prod_j p_\sigma^{(j)}$,

$$\alpha_\sigma p_\sigma = \alpha_\tau \delta_{\sigma\tau}^{*} p_\tau = \alpha_\sigma (\bar{\delta}_{\sigma\tau}^{*} - \hat{u}_\sigma^{*} \hat{u}_\tau^{*}) \, p_\tau$$

$$= \hat{i}_\sigma^{*} \alpha_\sigma \Pi_1 + \hat{j}_\sigma^{*} \alpha_\sigma \Pi_2 + \hat{k}_\sigma^{*} \alpha_\sigma \Pi_3 + \hat{u}_\sigma^{*} \alpha_\sigma \Pi_4$$

Therefore, one obtains the Maxwell-Boltzman distribution law from Gibb's formulation by setting

$$\alpha_\sigma p_\sigma = \alpha_\sigma' \Pi_\sigma \tag{81}$$

from which is obtained:

$$\alpha_\sigma' = \hat{e}_{\sigma j \rho}^{*} \alpha_\rho \quad ; \quad \hat{e}_{\sigma j \rho}^{*} = (\hat{i}_\rho^{*}, \ \hat{j}_\rho^{*}, \ \hat{k}_\rho^{*}, \ \hat{u}_\rho^{*})$$

Thus finally one has, according to (47):

$$\alpha_\sigma' = \frac{\hat{e}_{\sigma j \rho}^{*} u_\rho^{*}}{kT} = (0, \ 0, \ 0, \ -c/kT) \tag{82}$$

and

$$w = \frac{e^{\frac{\hat{e}_{\sigma j \rho}^{*} u_\rho^{*}}{kT}}}{\int e^{\frac{\hat{e}_{\sigma j \rho}^{*} u_\rho^{*}}{kT}} d\phi} = \frac{e^{-\epsilon'/kT}}{\int e^{-\epsilon'/kT} d\phi}$$

in agreement with (49).

It should be noticed once again that, in (81), the left-hand-side of the equation is an inner product of two four vectors, whereas the right side is merely a sum of products of scalar quantities.

The average values can be obtained in this formulation by a procedure similar to that previously discussed in connection with the Maxwell-Boltzman distribution law.

MULTIPLE-TIME FORMULATION

The preceding covariant formulation of Gibb's statistical mechanics can be termed as a single time, t', formulation. The main feature of the formulation is that it involves only invariant quantities, and does not contain the space and time coordinates in a symmetrical manner. In a multiple-time formulation, one should assign to each component system its own individual space-time coordinates $x_\sigma^{(j)} = (x_1^{(j)}, x_2^{(j)}, x_3^{(j)}, x_4^{(j)} = ict^{(j)})$.

With respect to the orthonormalized unit vectors given by (3); namely:

$$\hat{e}_{\alpha\sigma}^{\Delta} = (\hat{i}_\sigma^{\Delta}, \hat{j}_\sigma^{\Delta}, \hat{k}_\sigma^{\Delta}, \hat{u}_\sigma^{\Delta})$$

the position 4 - vector of a clock at rest in the secondary frame can be established by:

$$x_\sigma^{\Delta} = (x_1^{\Delta}, x_2^{\Delta}, x_3^{\Delta}, x_4^{\Delta} = ic\tau^{\Delta}) \tag{84}$$

leading to the position 4 - vector in the primary frame:

$$x_\sigma^{*} = L_{\sigma\tau}^{*} x_\tau^{\Delta} \tag{85}$$

Now then the scalar invariant quantity:

$$\tau^{\Delta} = -\hat{u}_\sigma^{*} x_\sigma^{*}/c = -\hat{u}_\sigma^{\Delta} x_\sigma^{\Delta}/c = -\frac{i}{c} x_4^{\Delta} \tag{86}$$

is equal to the time read on a clock at rest in the secondary frame.

This procedure is identical to the one leading to obtaining the proper time in the rest frame. Indeed when one measures a quantity at rest in the secondary frame, there exists no essential difference between the secondary

frame and the rest frames. Therefore one may set:

$$d\tau^{(j)} = -\hat{u}_\sigma^{(j)} dx_\sigma^{(j)}/c = -\hat{u}_\sigma^{1} dx_\sigma^{1}/c = d\tau^{0} \qquad (j = 1, 2, 3, ---) \tag{87}$$

where $\tau^{(j)}$ is the proper time for the j-th particle. This indicates the fact that the rate of flow of time is the same with respect to any reference frame when measured by a clock at rest in that frame. One will, however, distinguish $d\tau^{(j)}$ from $d\tau^{0}$ for a while, though both are essentially the same.

It is known that the Lagrangian function does not have any particular Lorentz transformation property and thus one must expect it to be a world invariant scalar. The physically significant invariant Lagrangean of a system for the multiple-time formulation shall be given by: $\mathbb{L}(x_\sigma^{(j)}, u_\sigma^{(j)} \quad)$ where τ^{0} enters as an implicit parameter and $u_\sigma^{(j)}$ is defined by: $u_\sigma^{(j)} = \frac{dx_\sigma^{(j)}}{d\tau^{(j)}} = \frac{dx_\sigma^{(j)}}{d\tau^{0}}$, which is the velocity 4 - vectors as given by (10). It is noted here that the above defined Lagrangian is rheonomic, i.e. time dependent, and is in the most general form.

Now the Hamilton principle can be given by

$$\delta \int_{\tau_1^0}^{\tau_2^0} \mathbb{L}(x_\sigma^{(j)}, u_\sigma^{(j)})\, d\tau^{0} = 0 \tag{88}$$

the Lagrangian equations of motion for a system of ideal gas particles can be derived from it, as usual, leading to:

$$\frac{d}{d\tau^{0}}\left(\frac{\partial L}{\partial u_\sigma^{(j)}}\right) - \frac{\partial L}{\partial x_\sigma^{(j)}} = 0 \tag{89}$$

The canonically conjugate momentum can be defined by:

$$\frac{\partial L}{\partial u_\sigma^{(j)}} = p_\sigma^{(j)} \tag{90}$$

In this system, the total components of energy-momentum 4 - vector should be conserved leading to:

$$\frac{d}{d\tau^0}\Big(\sum_j \frac{\partial L}{\partial u_\sigma^{(j)}}\Big) = \frac{d}{d\tau^0}\Big(\sum_j p_\sigma^{(j)}\Big) = 0 \qquad (\sigma = 1, 2, 3, 4)$$

Thus from (89):

$$\sum_j \frac{\partial L}{\partial x_\sigma^{(j)}} = 0 \tag{91}$$

Since $p_\sigma^{(j)} = m_0^{(j)} u_\sigma^{(j)}$, with (90) and (91), one can evaluate:

$$\begin{aligned} L &= \sum_j \frac{m_0^{(j)}}{2} u_\sigma^{(j)} u_\sigma^{(j)} - V(x_\sigma^{(i)} - x_\sigma^{(j)}) \\ &= \sum_j \frac{p_\sigma^{(j)} p_\sigma^{(j)}}{2 m_0^{(j)}} - V(x_\sigma^{(i)} - x_\sigma^{(j)}) \end{aligned} \tag{92}$$

where V is an invariant function which depends only on the relative distances of the particles. Here again V shall only be appreciable instantaneously at the limits of $x_\sigma^{(i)} - x_\sigma^{(j)}$, and otherwise vanishes. Thus one can again assign the same role to V $(x_\sigma^{(i)} - x_\sigma^{(j)})$ as was done to V $(\xi_\varkappa^{(i)} - \xi_\varkappa^{(j)})$ in the single-time formulation. That is, V shall only control the instantaneous collision process, and assuming the interacting intervals for collisions to be so short that the part of L, namely:

$$\sum_j \frac{p_\sigma^{(j)} p_\sigma^{(j)}}{2 m_0^{(j)}}$$

itself can, at any instant, express the complete invariant Lagrangian of the system. In other words, one again keeps V in the Lagrangian L, just to indicate the fact that the particles are not completely free but execute collisions between themselves, changing instantaneously from one free motion state to another.

Now the Hamiltonian is defined by:

$$
\begin{aligned}
H &= \sum_j p_\sigma^{(j)} u_\sigma^{(j)} - L \\
&= \sum_j \frac{p_\sigma^{(j)} p_\sigma^{(j)}}{m_0^{(j)}} - \sum_j \frac{p_\sigma^{(j)} p_\sigma^{(j)}}{2m_0^{(j)}} + V\left(x_\sigma^{(i)} - x_\sigma^{(j)}\right) \\
&= \sum_j \frac{p_\sigma^{(j)} p_\sigma^{(j)}}{2m_0^{(j)}} + V\left(x_\sigma^{(i)} - x_\sigma^{(j)}\right)
\end{aligned}
\tag{93}
$$

The corresponding Hamilton equations of motion are:

$$
\frac{\partial H}{\partial p_\sigma^{(j)}} = \frac{dx_\sigma^{(j)}}{d\tau^\Delta}, \quad \frac{\partial H}{\partial x_\sigma^{(j)}} = -\frac{dp_\sigma^{(j)}}{d\tau^\Delta}
$$

or

$$
\frac{\partial H}{\partial p_\sigma^{(j)}} = \frac{dx_\sigma^{(j)}}{d\tau^{(j)}}, \quad \frac{\partial H}{\partial x_\sigma^{(j)}} = -\frac{dp_\sigma^{(j)}}{d\tau^{(j)}} \tag{94}
$$

Now assuming that each component of $x_\sigma^{(j)}$ and $p_\sigma^{(j)}$ are completely independent, one defines the phase space by introducing space-time and energy-momentum coordinates into a single coordinate organization for an ensemble. If there exist n component systems, therefore, the total number of degrees of freedom of the resultant system is then 4n and one shall consider 8n-dimensional phase space.

Once again introducing the invariant phase density function $\rho\ (x_\sigma^{(j)}, p_\sigma^{(j)})$ where τ^Δ is involved as an implicit parameter, the phase current vector can be defined by:

$$
D = \left(D_{x_\sigma^{(j)}},\ D_{p_\sigma^{(j)}} \right) \tag{95}
$$

where

$$
D_{x_\sigma^{(j)}} = \rho\, \dot{x}_\sigma^{(j)}, \quad D_{p_\sigma^{(j)}} = \rho\, \dot{p}_\sigma^{(j)}; \quad \dot{x}_\sigma^{(j)} = \frac{dx_\sigma^{(j)}}{d\tau^\Delta}, \ \dot{p}_\sigma^{(j)} = \frac{dp_\sigma^{(j)}}{d\tau^\Delta} \tag{96}
$$

One assumes the existence of the equation of continuity in 8n-dimensional phase-space:

$$\operatorname{div} \underline{D} = \sum_j \left(\frac{\partial P_{x_\sigma^{(j)}}}{\partial x_\sigma^{(j)}} + \frac{\partial P_{p_\sigma^{(j)}}}{\partial p_\sigma^{(j)}} \right) = 0 \tag{97}$$

Using the Hamilton equations of motion (94), it can be written by:

$$\operatorname{div} \underline{D} = [\rho, H]_{8n} = 0 \tag{97)'$$

where

$$[\rho, H]_{8n} = \sum_j \left[\frac{\partial \rho}{\partial x_\sigma^{(j)}} \frac{\partial H}{\partial p_\sigma^{(j)}} - \frac{\partial \rho}{\partial p_\sigma^{(j)}} \frac{\partial H}{\partial x_\sigma^{(j)}} \right] \tag{98}$$

which indicates the Poisson's bracket in the ($x_\sigma^{(j)}$, $p_\sigma^{(j)}$) 8n-dimensional phase space.

To see the physical significance of the 8n-dimensional equation of continuity, consider the fact that actually $p_\sigma^{(j)}$ are not completely independent but they are related by a relation, namely:

$$p_\sigma^{(j)} p_\sigma^{(j)} = \sum_{\varkappa=1}^{3} p_\varkappa^{(j)2} + p_4^{(j)2} = \sum_{\varkappa=1}^{3} p_\varkappa^{(j)2} - \left(\frac{E^{(j)}}{c}\right)^2 = -\left(\frac{E_0^{(j)}}{c}\right)^2$$

$$p_4^{(j)} = \frac{i E^{(j)}}{c} = i \sqrt{\sum_{\varkappa=1}^{3} p_\varkappa^{(j)2} + (m_0^{(j)} c)^2}$$

of simply:

$$p_4^{(j)} = p_4^{(j)} (p_\varkappa^{(j)}) \tag{99}$$

$$\frac{d p_4^{(j)}}{d z^0} = \sum_{\varkappa=1}^{3} \frac{\partial p_4^{(j)}}{\partial p_\varkappa^{(j)}} \frac{d p_\varkappa^{(j)}}{d z^0} \tag{99)'$$

By imposing the functionality (99), with some rearrangement, (98) can be shown to yield:

$$\operatorname{div} \underline{D} = \sum_j \left(\frac{\partial \rho}{\partial x_\sigma^{(j)}} \frac{d x_\sigma^{(j)}}{d z^0} + \frac{\partial \rho}{\partial p_\sigma^{(j)}} \frac{d p_\sigma^{(j)}}{d z^0} \right)$$

$$= \sum_j \left(\frac{\partial \rho}{\partial t^{(j)}} \frac{dt^{(j)}}{d\tau^o} + \sum_{\varkappa=1}^{3} \frac{\partial \rho}{\partial x_\varkappa^{(j)}} \frac{dx_\varkappa^{(j)}}{d\tau^o} + \sum_{\varkappa=1}^{3} \frac{\partial \rho}{\partial p_\varkappa^{(j)}} \frac{dp_\varkappa^{(j)}}{d\tau^o} + \frac{\partial \rho}{\partial p_4^{(j)}} \frac{dp_4^{(j)}}{d\tau^o} \right)$$

$$= \sum_j \left(\frac{\partial \rho}{\partial t^{(j)}} \frac{dt^{(j)}}{d\tau^o} + \sum_{\varkappa=1}^{3} \frac{\partial \rho}{\partial x_\varkappa^{(j)}} \frac{dx_\varkappa^{(j)}}{d\tau^o} + \sum_{\varkappa=1}^{3} \frac{\partial \rho}{\partial p_\varkappa^{(j)}} \frac{dp_\varkappa^{(j)}}{d\tau^o} + \frac{\partial \rho}{\partial p_4^{(j)}} \frac{dp_4^{(j)}}{d\tau^o} \right)$$

$$= \sum_j \left(\frac{\partial \rho}{\partial t^{(j)}} \frac{dt^{(j)}}{d\tau^o} \right) + \sum_j \sum_\varkappa \left[\frac{\partial \rho}{\partial x_\varkappa^{(j)}} \frac{dx_\varkappa^{(j)}}{d\tau^o} + \frac{\partial \rho}{\partial p_\varkappa^{(j)}} \frac{dp_\varkappa^{(j)}}{d\tau^o} \right] = 0$$

(100)

Again noting the functionality of $\rho\,(x_\sigma^{(j)}, p_\sigma^{(j)})$ with τ^o as an implicit parameter, one writes:

$$\sum_j \frac{\partial \rho}{\partial t^{(j)}} \frac{dt^{(j)}}{d\tau^o} = \frac{\delta \rho}{\delta \tau^o} \qquad (101)$$

keeping in mind that $\frac{\delta \rho}{\delta \tau^o}$ indicates a partial derivative of ρ with respect to parameter τ^o, only varying $t^{(j)}$ and fixing all the other variables as constants. (100) can then be written as:

$$\operatorname{div} \underline{D} = \frac{\delta \rho}{\delta \tau^o} + [\rho, H]_{6n} = 0 \qquad (102)$$

where $[\rho, H]_{6n} = \sum_j^n \sum_\varkappa^3 \left(\frac{\partial \rho}{\partial x_\varkappa^{(j)}} \frac{\partial H}{\partial p_\varkappa^{(j)}} - \frac{\partial \rho}{\partial p_\varkappa^{(j)}} \frac{\partial \rho}{\partial x_\varkappa^{(j)}} \right)$ which indicates the corresponding Poisson's bracket in ($x_\varkappa^{(j)}$, $p_\varkappa^{(j)}$) 6n - dimensional phase space. It is seen that (102) yields the identical form of Liouville's theorem with the single time formulation in the 6n-phase space, if one lets $\frac{\delta \rho}{\delta \tau^o}$ correspond to $\frac{\partial \rho}{\partial t'}$ and $[\rho, H]_{6n}$ to $[\rho, H]$ in (68).

The above fact can be interpreted as follows: One may consider that each component of $x_\sigma^{(j)}$ and $p_\sigma^{(j)}$ are completely independent and thus may define the trivial 8n - dimensional phase space ($x_\sigma^{(j)}$, $p_\sigma^{(j)}$). Once this

procedure is achieved, one can always reduce it to the corresponding 6n - dimensional phase space formulation by imposing the condition (98). There exists an advantage in this procedure. Since the 8n - dimensional formulation (97) is completely covariant in form, if one can impose the condition (99) in the relativistic covariant form, one can be assured of a completely covariant formulation leading to the corresponding results obtained by the ordinary 6n - dimensional formulation. Later this procedure will be seen to be possible by introducing an additional covariant Δ -function.

It is noted here that (97)', $[\rho, H]_{8n} = 0$, is not the condition for a stationary state, but is a general condition to be obeyed by ρ. The stationary state condition can be, however, defined from (102) by an additional condition, namely:

$$\frac{\partial \rho}{\partial \tau_0} = 0 \tag{103}$$

Therefore for the equilibrium distribution function, one solves (97)' and (103) simultaneously. It is again seen that they can simultaneously be satisfied by choosing the world invariant phase density function ρ to be a function of constants of the motion of a system. Further, as it was done before, one may again assume ρ to be a function of total components of energy-momentum vector of our isolated statistical system, leading to:

$$\rho = \rho \quad (p_1, p_2, p_3, p_4 = i\frac{E}{c}) \tag{104}$$

where: $\sum_j p_\sigma^{(j)} = p_\sigma \quad , \quad \sum_j E^{(j)} = E \qquad (\sigma = 1, 2, 3, 4)$

The phase probability coefficient $w = \rho/n$ of a whole system can be connected with the component phase probability coefficients $w_j = w_j \; (p_1^{(j)}, \cdots, p_4^{(j)})$ which can be defined with the similar assumption as in the single-time formulation by:

$$\prod_{j=1}^{n} w_j\left(p_1^{(j)}, \cdots p_4^{(j)}\right) = w\left(p_1, \cdots, p_4\right) \tag{105}$$

(105) is identical with (76) in form, thus it can be led to a relation similar to (78) in form, namely:

$$\alpha_\sigma'' = \frac{1}{w_i}\frac{\partial w_i}{\partial p_\sigma^{(i)}} = \frac{1}{w_j}\frac{\partial w_j}{\partial p_\sigma^{(j)}} = \frac{1}{w}\frac{\partial w}{\partial p_\sigma} \text{ (a constant)} \tag{106}$$

with its solution:

$$\log w_j = \alpha_\sigma'' p_\sigma^{(j)} + \gamma^{(j)}$$

$$w_j = e^{\gamma(j)} e^{\alpha_\sigma'' p_\sigma^{(j)}}$$

$$W = \prod_j w_j = e^{\Gamma} e^{\sum_j \alpha_\sigma'' p_\sigma^{(j)}} = e^{\Gamma} e^{\alpha_\sigma'' p_\sigma} \tag{107}$$

where $\Gamma = \sum_j \gamma^{(j)}, \quad p_\sigma = \sum_j p_\sigma^{(j)}$

Now the volume element in the phase space can be written:

$$d\phi = \prod_{j=1}^{4} \prod_{\alpha=1}^{4} dx_\alpha^{(j)}\, dp_\sigma^{(j)} \tag{108}$$

It is tempting in this formulation to set $\int w\, d\phi = 1$. However, in equilibrium state, no statistical quantity is to be explicitly dependent either on the relative time $t^{(i)}$ or on the proper time τ^{o}. Therefore the integration over the time in $\int w\, d\phi$ is essentially meaningless. Further the condition (99) is to be imposed in a covariant form. The restriction (99) and the elimination of the trivial time factor can be carried out in a relativistic manner by adding to the phase volume element $d\phi$ the invariant factor Δ, which is defined by:

$$\Delta = \prod_{j=1}^{n} \frac{2E^{(j)}/c}{d(ict^{(j)})}\, \delta\left(p_\sigma^{(j)} p_\sigma^{(j)} + \left(\frac{E_0^{(j)}}{c}\right)^2\right) \tag{109}$$

where $\delta(x)$ is the Dirac delta function.

The following relation holds for the Dirac delta function:

$$\delta\left((x-x_1)(x-x_2)\right) = \frac{\delta(x-x_1)+\delta(x-x_2)}{|x_1-x_2|}$$

Now $\delta\left(p_\sigma^{(j)} p_\sigma^{(j)} + (E_0^{(j)}/c)^2\right)$ can then be written as:

$$\delta\left(p_\sigma^{(j)} p_\sigma^{(j)} + (E_0^{(j)}/c)^2\right) = \delta\left(p^{(j)2} - (E^{(j)}/c)^2 + (E_0^{(j)}/c)^2\right)$$

$$= \delta\left(p^{(j)2} + (m_0^{(j)}c)^2 - (E^{(j)}/c)^2\right)$$

$$= \delta\left(\left[\sqrt{p^{(j)2} + (m_0^{(j)}c)^2}\right]^2 - (E^{(j)}/c)^2\right)$$

$$= \delta\left(\left[\sqrt{p^{(j)2} + (m_0^{(j)}c)^2} + E^{(j)}/c\right]\left[\sqrt{p^{(j)2} + (m_0^{(j)}c)^2} - E^{(j)}/c\right]\right)$$

$$= \frac{\delta\left(\sqrt{p^{(j)2} + (m_0^{(j)}c)^2} + E^{(j)}/c\right) + \delta\left(\sqrt{p^{(j)2} + (m_0^{(j)}c)^2} - E^{(j)}/c\right)}{2E^{(j)}/c}$$

$$= \frac{\delta\left(\varepsilon^{(j)} + E^{(j)}/c\right) + \delta\left(\varepsilon^{(j)} - E^{(j)}/c\right)}{2E^{(j)}/c}$$

$$\therefore \Delta = \prod_j \frac{2E^{(j)}/c}{dx_4^{(j)}}\,\delta\left(p_\sigma^{(j)} p_\sigma^{(j)} + (E_0^{(j)}/c)^2\right)$$

$$= \prod_j 1/dx_4^{(j)}\left[\delta\left(\varepsilon^{(j)} + E^{(j)}/c\right) + \delta\left(\varepsilon^{(j)} - E^{(j)}/c\right)\right]$$

where $\varepsilon^{(j)} = \sqrt{p^{(j)2} + (m_0^{(j)}c)^2}$

restricting oneself to $\varepsilon^{(j)} > 0$ only.

Now with this Δ- function, the phase probability coefficient W must satisfy:

$$\int W\,d\Phi = 1\,, \qquad d\Phi = \Delta\,d\phi \tag{111}$$

Then from (107):

$$e^{-T} = \int e^{\alpha_\sigma'' p_\sigma} \, d\Phi = Z \qquad \text{(partition function)} \tag{112}$$

$$w\,d\Phi = \frac{e^{\alpha_\sigma'' p_\sigma} \, d\Phi}{\int e^{\alpha_\sigma'' p_\sigma} \, d\Phi}. \tag{113}$$

In analogy with the case of the single time formulation, comparing with the result obtained in the Maxwell-Boltzman distribution law, one sets:

$$\alpha_\sigma'' = \alpha_\sigma = U_\sigma^{*}/kT \tag{114}$$

$$w\,d\Phi = \frac{e^{\alpha_\sigma p_\sigma} \, d\Phi}{\int e^{\alpha_\sigma p_\sigma} \, d\Phi}$$

$$= \frac{e^{\alpha_\sigma p_\sigma} \prod_{j=1}^{n} \left\{ \frac{1}{dx_4^{(j)}} \left[\delta\left(\xi^{(j)} + E^{(j)}/c\right) + \delta\left(\xi^{(j)} - E^{(j)}/c\right) \right] \left[\prod_{\alpha=1}^{4} dx_\alpha^{(j)} \, dp_\alpha^{(j)} \right] \right\}}{\int e^{\alpha_\sigma p_\sigma} \prod_{j=1}^{n} \left\{ \frac{1}{dx_4^{(j)}} \left[\delta\left(\xi^{(j)} + E^{(j)}/c\right) + \delta\left(\xi^{(j)} - E^{(j)}/c\right) \right] \left[\prod_{\alpha=1}^{4} dx_\alpha^{(j)} \, dp_\alpha^{(j)} \right] \right\}}$$

Neglecting the part of $\delta\left(\xi^{(j)} + E^{(j)}/c\right)$, it yields:

$$w\,d\Phi = \frac{e^{\alpha_\sigma p_\sigma} \prod_{j=1}^{n} \left\{ \delta\left(\xi^{(j)} - E^{(j)}/c\right) d\left(E^{(j)}/c\right) \prod_{\varkappa=1}^{3} dx_\varkappa^{(j)} \, dp_\varkappa^{(j)} \right\}}{\int e^{\alpha_\sigma p_\sigma} \prod_{j=1}^{n} \left\{ \delta\left(\xi^{(j)} - E^{(j)}/c\right) d\left(E^{(j)}/c\right) \prod_{\varkappa=1}^{3} dx_\varkappa^{(j)} \, dp_\varkappa^{(j)} \right\}} \tag{115}$$

Again to be specific, restricting oneself to an ensemble for a single particle system, one has:

$$w\,d\Phi = \frac{e^{\alpha_\sigma p_\sigma} \delta\left(\xi - E/c\right) d\left(E/c\right) \prod_{\varkappa=1}^{3} dx_\varkappa \, dp_\varkappa}{\int e^{\alpha_\sigma p_\sigma} \delta\left(\xi - E/c\right) d\left(E/c\right) \prod_{\varkappa=1}^{3} dx_\varkappa \, dp_\varkappa}$$

For a physical quantity A_σ, its average value is given by:

$$\overline{A_\sigma} = \overline{\mathrm{L}^{x}_{\sigma\rho} A'_\rho} = \mathrm{L}^{\lambda}_{\sigma\rho}\, \overline{A'_\rho}$$

where:

$$\overline{A'_\rho} = \frac{\int A'_\rho\, e^{\alpha_\sigma p_\sigma}\, \delta(\varepsilon - E/c)\, d(E/c) \prod_{\varkappa=1}^{3} dx_\varkappa\, dp_\varkappa}{\int e^{\alpha_\sigma p_\sigma}\, \delta(\varepsilon - E/c)\, d(E/c) \prod_{\varkappa=1}^{3} dx_\varkappa\, dp_\varkappa}$$

$$= \frac{\int A'_\rho\, e^{-E'/kT}\, \delta(\varepsilon' - E'/c)\, d(E'/c) \prod_{\varkappa=1}^{3} dx'_\varkappa\, dp'_\varkappa}{\int e^{-E'/kT}\, \delta(\varepsilon' - E'/c)\, d(E'/c) \prod_{\varkappa=1}^{3} dx'_\varkappa\, dp'_\varkappa}$$

$$= \frac{\int A'_\rho\, e^{-c\varepsilon'/kT} \prod_{\varkappa=1}^{3} dx'_\varkappa\, dp'_\varkappa}{\int e^{-c\varepsilon'/kT} \prod_{\varkappa=1}^{3} dx'_\varkappa\, dp'_\varkappa} \qquad (116)$$

where $\varepsilon' = \sqrt{p^2 + (m_0 c)^2}$. Remembering that it is essentially:

$$x'_\varkappa = \xi_\varkappa, \qquad p'_\varkappa = \Pi_\varkappa \qquad (\varkappa = 1, 2, 3)$$

it is seen that (116) is exactly in agreement with (51), a result of the Maxwell-Boltzman distribution law. In deriving (51), the 6 - dimensional phase space was rather arbitrarily introduced, but the above agreement is seen to be a logical verification of its validity.

CONCLUSIONS

The conservation law of the four energy-momentum vector components of a whole isolated statistical system is utilized in formulating the covariant statistical mechanics. The three reference frames: rest, secondary, and

primary, are helpful devices in this discussion.

First the expansion of the Maxwell-Boltzman distribution law is discussed and found to yield the result:

$$w_j = \frac{g_i\, e^{\alpha_\sigma p_\sigma^{(j)}}}{\sum_j g_i\, e^{\alpha_\sigma p_\sigma^{(j)}}}$$

where:

$$\alpha_\sigma = u_\sigma^{*}/kT.$$

To formulate the single-time Gibb's statistical mechanics, the equations of motion are set up with invariant quantities only, and it leads to the covariant Liouville's theorem:

$$[\rho, H] + \frac{\partial \rho}{\partial t'} = 0$$

with the stationary condition $\frac{\partial \rho}{\partial t'} = 0$. This is identical with the classical formulation in form, but is not essentially the same.

One is led to the canonical distribution law:

$$w = \frac{e^{\alpha_\sigma' \pi_\sigma}}{\int e^{\alpha_\sigma' \pi_\sigma}\, d\phi}$$

with $\quad \alpha_\sigma' = \hat{e}_{\sigma\rho}^{*} u_\rho^{*}/kT \quad , \quad \hat{e}_{\alpha\rho}^{*} = (\hat{i}_\rho^{*}, \hat{j}_\rho^{*}, \hat{k}_\rho^{*}, \hat{u}_\rho^{*})$

Further the multiple-time formulation is considered with the result:

$$w\, d\Phi = \frac{e^{\alpha_\sigma p_\sigma} \prod_{j=1}^{N} \left\{ \delta\left(\varepsilon^{(j)} - E^{(j)}/c\right) d\left(E^{(j)}/c\right) \prod_{\varkappa=1}^{3} dx_\varkappa^{(j)}\, dp_\varkappa^{(j)} \right\}}{\int e^{\alpha_\sigma p_\sigma} \prod_{j=1}^{N} \left\{ \delta\left(\varepsilon^{(j)} - E^{(j)}/c\right) d\left(E^{(j)}/c\right) \prod_{\varkappa=1}^{3} dx_\varkappa^{(j)}\, dp_\varkappa^{(j)} \right\}}$$

with

$$\varepsilon^{(j)} = \sqrt{p^{(j)2} + (m_0^{(j)} c)^2} > 0 \quad , \quad d\Phi = \delta\, d\phi.$$

$$\Delta = \prod_{j=1}^{n} \frac{1}{dx_4^{(j)}} \left\{ \delta\left(\varepsilon^{(j)}_4 + E^{(j)}/c\right) + \delta\left(\varepsilon^{(j)} - E^{(j)}/c\right) \right\}$$

This formulation is based on the covariant Liouville's theorem:

$$[\rho, H]_{8n} = 0 .$$

with the stationary condition:

$$\frac{\partial \rho}{\partial \tau^0} = 0 .$$

In any of the above formulations, the phenomenological relativistic distribution law appears to be an approximation at low relative velocity u_σ^{τ}, and the quantized statistical law can now be obtained simply by replacing the variables by their appropriate operators. Indeed the Krotkov and Scheidegger's quantized statistical law can be obtained from (28) by simply replacing the momentum $p_\sigma^{(j)}$ with its operator in the position representation, and by making the appropriate change of notations.

ACKNOWLEDGMENT

The author wishes to acknowledge with sincere gratitude and appreciation the excellent advice and helpful criticisms throughout this research of Dr. Boris Leaf under whose patient guidance this work was done. He would also like to express his sincere appreciation to Dr. S. E. Whitcomb, head of Physics Department, and Dr. A. B. Cardwell, director of General Research, for their constant encouragement and support extended to him. Finally professors and graduate students in Physics Department, especially Dr. R. H. McFarland, Dr. Basil Curnutte, Dr. Robert Katz and Dr. C. M. Fowler, are gratefully acknowledged for their kind guidance, assistance and friendship.

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THE FORMULATION OF THE RELATIVISTIC
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by

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B. S., Chosun University, 1952
Kwangju, Korea

AN ABSTRACT OF A THESIS

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Physics

KANSAS STATE COLLEGE
OF AGRICULTURE AND APPLIED SCIENCE

1957

The conservation law for the four components of the energy-momentum vector of a whole isolated statistical system is utilized in formulating the covariant statistical mechanics. Three reference frames, rest, secondary and primary, are helpful devices in this discussion.

Unit vectors at rest in the secondary frame are defined by:

$$\hat{e}^{\triangle}_{\alpha j\sigma} = (\hat{i}^{\triangle}_{\sigma}, \hat{j}^{\triangle}_{\sigma}, \hat{k}^{\triangle}_{\sigma}, \hat{u}^{\triangle}_{\sigma})$$

$$\hat{i}^{\triangle}_{\sigma} = (1, 0, 0, 0)$$

$$\hat{j}^{\triangle}_{\sigma} = (0, 1, 0, 0)$$

$$\hat{k}^{\triangle}_{\sigma} = (0, 0, 1, 0)$$

$$\hat{u}^{\triangle}_{\sigma} = (0, 0, 0, i)$$

The orthogonal unit vectors in the primary frame $\hat{e}^{*}_{\alpha j\sigma} = (\hat{i}^{*}_{\sigma}, \hat{j}^{*}_{\sigma}, \hat{k}^{*}_{\sigma}, \hat{u}^{*}_{\sigma})$ are then given by:

$$\hat{e}^{*}_{\alpha j\sigma} = L^{*}_{\sigma z}\,\hat{e}^{\triangle}_{\alpha j z}$$

A similar definition is given for the orthogonal unit vectors transforming from the rest to the primary frame by:

$$\hat{e}^{(j)}_{\alpha j\sigma} = L^{(j)}_{\sigma z}\,\hat{e}^{\circ}_{\alpha j z}$$

where

$$\hat{e}^{\circ}_{\alpha j\sigma} = (\hat{i}^{\circ}_{\sigma}, \hat{j}^{\circ}_{\sigma}, \hat{k}^{\circ}_{\sigma}, \hat{u}^{\circ}_{\sigma}) \; ; \; \hat{e}^{(j)}_{\alpha j\sigma} = (\hat{i}^{(j)}_{\sigma}, \hat{j}^{(j)}_{\sigma}, \hat{k}^{(j)}_{\sigma}, \hat{u}^{(j)}_{\sigma})$$

and $\hat{e}^{\circ}_{\alpha j\sigma}$ is essentially the same as $\hat{e}^{\triangle}_{\alpha j\sigma}$ and $L^{(j)}_{\sigma z}$ is the Lorentz transformation matrix from j-th rest frame to the primary frame.

With position 4 - vector $x^{(j)}_{\sigma} = (x^{(j)}_1, x^{(j)}_2, x^{(j)}_3, x^{(j)}_4 = ict^{(j)})$, the following invariant quantities are defined:

$$-\frac{\hat{u}^{*}_{\sigma}\, x^{(j)}_{\sigma}}{c} = t', \qquad \xi^{(j)}_{\varkappa} = \hat{e}^{*}_{\varkappa j\sigma}\, x^{(j)}_{\sigma}$$

$$\begin{pmatrix} \varkappa = 1, 2, 3 \\ j = 1, 2, 3, \text{-----} \end{pmatrix}$$

where t' is an invariant parameter equal to the time as read in the secondary frame and $\xi_{\varkappa}^{(j)}$ is equal in magnitude to the Cartesian components of position as measured relative to the secondary frame. Further with the energy-momentum 4 - vector $p_{\sigma}^{(j)} = m_0^{(j)} u_{\sigma}^{(j)}$, the invariant quantities which are equal in magnitude to the Cartesian components of momentum and energy of the particles relative to the secondary frame, are defined by:

$$\Pi_{\varkappa}^{(j)} = \hat{e}_{\varkappa J_\sigma}^{\ast} \, p_{\sigma}^{(j)} \qquad (\varkappa = 1, 2, 3)$$
$$\Pi_{4}^{(j)} = -\hat{u}_{\sigma}^{\ast} \, p_{\sigma}^{(j)}$$

To formulate the Maxwell-Boltzman distribution law in a covariant form, one must consider a property of a system as associated with a set of k boxes with a definite value of the property attached to each box. The elements are assumed to be indistinguishable in nature and move freely except executing any kind of collision between elements. Considering the maximum probable distribution, one obtains:

$$w_j = \frac{g_j \, e^{\alpha_\sigma p_\sigma^{(j)}}}{\sum_j g_j \, e^{\alpha_\sigma p_\sigma^{(j)}}} \quad ; \qquad \sum_j w_j = 1$$

where g_j is the apriori probability related to the j-th box.

$p_{\sigma}^{(j)}$ is the energy-momentum 4 - vector related to the j-th box, and thus w_j is the probability that an element may be found in the j-th box. The properties of 4 - vector α_σ are discussed yielding:

$$\alpha_\sigma = u_\sigma^{\ast}/kT$$

where $u_\sigma^{\ast}$ is the velocity 4 - vector of the secondary frame as seen by a primary observer.

Gibb's single time, t', in relativistic covariant statistical mechanics is further considered in terms of such invariant quantities as $\Pi_{\varkappa}^{(j)}$,

$\dot{\xi}_{\varkappa}^{(j)\prime}$, t' and $\Pi_4^{(j)}$. Starting from the covariant Hamilton principle:

$$\delta I = \delta \int_{t_1'}^{t_2'} L\,(\xi_{\varkappa}^{(j)}, \dot{\xi}_{\varkappa}^{(j)}, t')\,dt' = 0$$

the Lagrangian equations of motion:

$$\frac{d}{dt'}\left(\frac{\partial L}{\partial \dot{\xi}_{\varkappa}^{(j)}}\right) - \frac{\partial L}{\partial \xi_{\varkappa}^{(j)}} = 0$$

and the Hamilton's equation of motion:

$$\frac{\partial H}{\partial \Pi_{\varkappa}^{(j)}} = \frac{d\xi_{\varkappa}^{(j)}}{dt'}, \quad \frac{\partial H}{\partial \xi_{\varkappa}^{(j)}} = -\frac{d\Pi_{\varkappa}^{(j)}}{dt'}$$

are derived with

the Lagrangean, $L = \sum_j \left(-m_0^{(j)} c^2 \sqrt{1 - \dot{\xi}^{(j)2}/c^2}\right) - V(\xi_{\varkappa}^{(i)} - \xi_{\varkappa}^{(j)})$

the Hamiltonian, $H = \sum_j \left(\dfrac{m_0^{(j)} c^2}{\sqrt{1 - \dot{\xi}^{(j)2}/c^2}}\right) + V(\xi_{\varkappa}^{(i)} - \xi_{\varkappa}^{(j)})$

The covariant Licuville's theorem:

$$[\rho, H] + \frac{\partial \rho}{\partial t'} = 0$$

is obtained with the stationary condition $\frac{\partial \rho}{\partial t'} = 0$, where ρ is the world invariant phase density function. This procedure is identical with the classical formulation in form, but it is seen that they are essentially different. Finally the canonical distribution law is given by:

$$W = \frac{e^{\alpha_\sigma' \Pi_\sigma}}{\int e^{\alpha_\sigma' \Pi_\sigma}\, d\phi}$$

where $\Pi_\sigma = \sum_j \Pi_\sigma^{(j)}$, $d\phi = \prod_j \prod_{\varkappa} d\xi_{\varkappa}^{(j)} d\Pi_{\varkappa}^{(j)}$, the phase volume element, and

$$\alpha_\sigma' \Pi_\sigma = \alpha_\sigma P_\sigma$$

Further the multiple-time formulation is considered with the result:

$$W\, d\Phi = \frac{e^{\alpha_\sigma p_\sigma} \prod_{j=1}^{n} \left[\delta\left(\xi^{(j)} - E^{(j)}/c\right) d\left(E^{(j)}/c\right) \prod_{\varkappa=1}^{3} dx_\varkappa^{(j)}\, dp_\varkappa^{(j)} \right]}{\int e^{\alpha_\sigma p_\sigma} \prod_{j=1}^{n} \left[\delta\left(\xi^{(j)} - E^{(j)}/c\right) d\left(E^{(j)}/c\right) \prod_{\varkappa=1}^{3} dx_\varkappa^{(j)}\, dp_\varkappa^{(j)} \right]}$$

with: $\xi^{(j)} = \sqrt{p^{(j)2} + (m_0^{(j)} c)^2} > 0, \quad d\Phi = \Delta\, d\phi$

$$\Delta = \prod_{j=1}^{n} 1/dx_4^{(j)} \left[\delta\left(\xi^{(j)} + E^{(j)}/c\right) + \delta\left(\xi^{(j)} - E^{(j)}/c\right) \right]$$

$$d\phi = \prod_{j=1}^{n} \prod_{\sigma=1}^{4} dx_\sigma^{(j)}\, dp_\sigma^{(j)}$$

These results are based on the relativistic covariant Liouville's theorem in multiple-time formulation:

$$[\rho, H]_{g_n} = 0$$

with the stationary condition:

$$\frac{\delta \rho}{\delta \tau_0} = \sum_j \frac{\partial \rho}{\partial t^{(j)}} \frac{d t^{(j)}}{d \tau_0} = 0$$

The Hamiltonian of this case is given by $H = \sum_j \frac{p_\sigma^{(j)} p_\sigma^{(j)}}{2 m_0^{(j)}} + V(x_\sigma^{(i)} - x_\sigma^{(j)})$

In any of the above formulations, the ordinary relativistic distribution law appears to be an approximation at low relative velocity v*, and the quantized statistical law can be obtained simply by replacing the variables by their appropriate operators and by change of notations.